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INTERACTION BETWEEN A STATIONARY VORTEX TUBE AND AN INFINITE PLANE

V. A. Budarin

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A law of pressure variation in a near-wall radially converging flow formed in a viscous contact between a free vortex tube and a plane is found analytically.

This problem has been investigated in many works using the Navier–Stokes equation. As a result of the solutions obtained it has been shown that near a plane the vortex rotation becomes slower, the centrifugal force decreases, and a radially converging near-wall flow originates due to the pressure drop. It is typical that the vortex ceases to rotate everywhere except for a narrow region involving the vortex axis. As is noted and substantiated in [1, 2], this flow pattern differs from the notions of boundary-layer theory and allows one to neglect the effect of peripheral velocity on a near-wall flow at a distance from the vortex axis.

These specific features of a secondary flow make it possible to consider it as a two-dimensional one in which only the section-mean velocity V and near-wall flow thickness δ change along the radius.

The same works showed that a numerical solution of this problem exists only at small Reynolds numbers, though, as is mentioned in [3], it is possible in principle to obtain a solution at any Re.

By virtue of the existing difficulties in solving this problem, another approach is of interest to account for forces of viscous friction and determine the pressure distribution on a streamlined surface. This approach is based on the force balance equation for a small annular portion of the near-wall flow.

To derive this equation we distinguish in the near-wall flow an annular portion with a width dr and a thickness δ that is at a distance r from the axis of the vortex tube (Fig. 1). Here we neglect the forces of flow friction against surrounding particles assuming them to be insignificant compared to friction against the plane.

Assuming the pressure forces affecting the distinguished portion to be counterbalanced by the forces of friction against the plane and the inertia force caused by flow acceleration as approaching the axis, we have

$$\delta F_{\Delta p} = \delta F_{\text{fr}} + \delta F_{\text{in}} , \tag{1}$$

where $\delta F_{\Delta p} = 2\pi\delta[(r + dr/2)(p + dp/2) - (r - dr/2)(p - dp/2)$ is the resultant of pressure forces affecting the distinguished portion; $\delta F_{\text{fr}} = \pi C_{\text{f}}[(r + dr/2)^2 - (r - dr/2)^2](\rho V^2/2)$ is the elementary force of friction against the plane; $\delta F_{\text{in}} = dma$ is the elementary inertia force, with $dm = \overline{n}\rho\delta[(r + dr/2)^2 - (r - dr/2)^2]$ being the mass of the distinguished annular portion.

Since the acceleration of the flow decreases with an increase in the radius, i.e., $a = -dV/d\tau$, and the time $d\tau = dr/V$, then $a = -V(dV/dr)$. Substituting these relations into Eq. (1) and neglecting the terms of the second order of smallness, we obtain a linear differential equation of the near-wall flow

$$\frac{dp}{dr} + \frac{p}{r} = C_{\text{f}} \frac{\rho V^2}{2\delta} - \rho V \frac{dV}{dr} . \tag{2}$$

As follows from [2, 4], in the problems of this class the mean flow velocity V is usually taken to be in inverse proportion to the distance from the coordinate origin, i.e., $V = V_2 r_2 / r$, where V_2 and r_2 are the section-mean velocity and the radius of, e.g., a narrow flow section II, which are assumed known and constant in a stationary flow. We take the boundary between the flow and the motionless surrounding medium to be described by an

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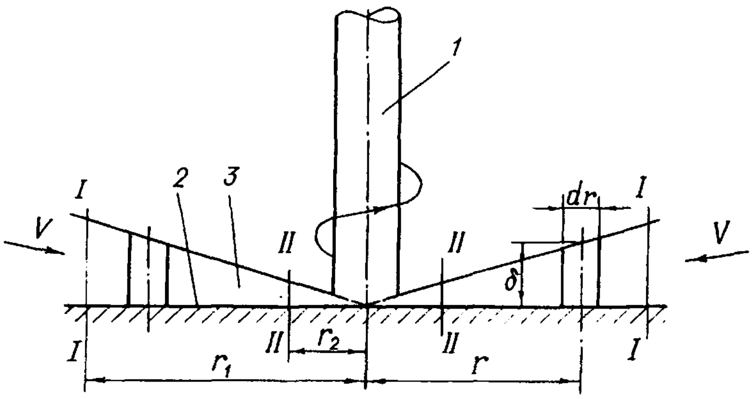

Fig. 1. Computational scheme of axisymmetric near-wall flow: 1) free vortex tube; 2) plane; 3) near-wall flow.

equation of a straight line in the form $\delta = br + c$ (where b and c are constant). Substituting the last equations into (2) we have

$$\frac{dp}{dr} + \frac{p}{r} = \frac{\rho V_2^2 r_2^2}{2}\left[\frac{C_f}{r^2(br+c)} + \frac{2}{r^3}\right]. \tag{3}$$

An equation similar to (2) or (3) can be obtained by calculating the elementary force of friction against the plane by the Newton law [5]. Then, $\delta T_{fr} = \pm \mu df(dV/d\delta)$. Comparing this expression for δT_{fr} with the previous one, we have

$$C_f = \frac{2\nu}{V^2}\frac{dV}{d\delta}.$$

Using the earlier assumptions about the change in the velocity along the radius ($Vr = V_2 r_2 = \text{const}$) and the linear relationship $\delta = f(r)$, we have

$$C_f = \frac{2\nu r^2}{V_2^2 r_2^2}\frac{dV/dr}{d\delta/dr} = \frac{\text{const}}{\text{Re}_2}.$$

It is seen that the effect of the Reynolds number in a radially converging flow should differ from that in a plane-parallel one, in which C_f is proportional to $\text{Re}^{-0.5}$.

Assuming the viscosity to be slightly variable, we can approximately consider C_f to be independent of the radius.

Using, e.g., the integrating multiplier $\mu(r) = \exp(\int dr/r) = r$ we obtain the solution of Eq. (3) in the form [6]

$$p = \frac{1}{r}\left[\int\left(\frac{\rho V_2^2 r_2^2}{2}\frac{C_f}{r^2(br+c)} + \frac{2}{r^3}\right) rdr + C_0\right],$$

and after transformations

$$p = \frac{C_0}{r} - \frac{\rho V_2^2 r_2^2}{2r}\left(\frac{C_f}{c}\ln\left|\frac{br+c}{r}\right| + \frac{2}{r}\right). \tag{4}$$

We consider a particular case of Eq. (3) which is based on the results of [1, 2], where it is shown that near the plane and at some distance from the vortex axis there exists a flow directed to the plane and the axis, with the radial velocity component exceeding the axial one. This allowed the authors to construct streamlines so that the thickness of the near-wall flow δ reduces with the radius, with streamlines leaving the plane on approaching the vortex axis.

Assuming the described qualitative character of the flow to be also preserved at large Reynolds numbers, we take in Eq. (3) $c = 0$. In this case (3) is reduced to the form

$$\frac{dp}{dr} + \frac{p}{r} = \frac{C_{\rho f}}{r^3}, \tag{5}$$

where $C_{\rho f} = (\rho V_2^2 r_2^2/2)(C_f/b + 2)$ is constant.

We consider some particular cases of the solution of Eq. (5). For ρ = const there takes place the solution with a general integral

$$p = -\frac{C_{\rho f}}{r^2} + \frac{C_0}{r}.$$

Since pressure cannot be negative, the condition of the origin of such near-wall flow in the form $C_0 > C_{\rho f}/r$ follows from the general solution.

For an ideal gas, which obeys the equation of state $p = \rho RT$, assuming the flow process to be polytropic p/ρ^n = const, we can find that the medium density ρ is related to the pressure in section II and in the current section by $\rho = p^{1/n} \rho_2^{(n-1)/n}/(RT_2)$.

Then Eq. (5) is reduced to the Bernoulli differential equation

$$\frac{dp}{dr} + \frac{p}{r} = C_{qf} \frac{p^{1/n}}{r^3} \tag{6}$$

where $C_{qf} = (V_2^2 r_2^2/2RT_2)p_2^{(n-1)/n}(C_f/b + 2)$ is constant.

The general integral of this equation has the form ($n \neq 1$)

$$p = \frac{1}{r}\left[-\frac{n-1}{n+1} C_{qf} r^{-\frac{n+1}{n}} + C_0\right]^{\frac{n-1}{n}}.$$

For an isothermally flowing ideal gas (5) acquires the form

$$\frac{dp}{dr} + \frac{p}{r} = C_{Tf} \frac{p}{r^3}, \tag{7}$$

where $C_{Tf} = (V_2^2 r_2^2/2RT)(C_f/b + 2)$ is constant.

The general integral of this equation is

$$p = \exp\left[-\frac{C_{Tf}}{2r^2} - \ln r + C_0\right].$$

An analysis of general integrals shows that physical and geometrical factors are the basic affecting ones that are interrelated and cannot be chosen arbitrarily. To estimate the effect of them we reduce the general integrals to dimensionless form using the boundary conditions for two annular flows lying on different radii (Fig. 1).

Then, taking $p = p_1$ at $r = r_1$ and $p = p_2$ at $r = r_2$, we obtain

for the case of ρ = const

$$\frac{p_1}{p_2} = \bar{r}^{-1} + C_\rho (\bar{r}^{-1} - \bar{r}^{-2}), \tag{8}$$

for a polytropic flow ($n \neq 1$)

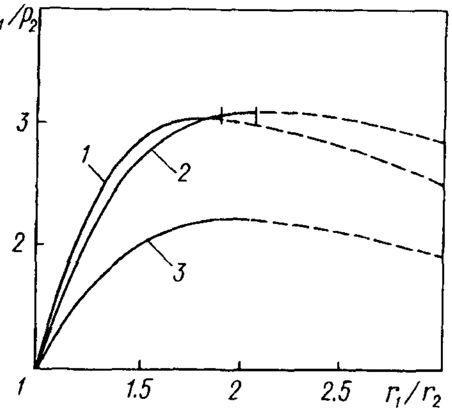

Fig. 2. Calculated distribution of pressure on a plane: 1) for an incompressible flow $C_p = 10$; 2) for a polytropic flow $C_q = 1$; 3) for an isothermal flow $C_T = 2$. Vertical dashes, position of section I; dashed curves, uncomputed region.

$$\frac{p_1}{p_2} = \left[C_q\left(\frac{1}{\bar{r}^{\frac{n-1}{n}}} - \bar{r}^{-2}\right) + \frac{1}{\bar{r}^{\frac{n-1}{n}}}\right]^{\frac{n}{n-1}}, \tag{9}$$

for an isothermal flow

$$\frac{p_1}{p_2} = \exp\left[C_T(1 - \bar{r}^{-2}) + \ln(\bar{r}^{-1})\right], \tag{10}$$

where $\bar{r} = r_1/r_2$ is the relative radius; $C_p = (\rho V_2^2/2p_2)(C_f/b + 2)$, $C_q = [(n-1)/(n+1)](V_2^2/RT_2)(C_f/b + 2)$, $C_T = (V_2^2/4RT)(C_f/b + 2)$ are dimensionless constants for corresponding flows that allow for the effect of physical factors only and involve known parameters in section II.

We consider the effect of the dimensionless constants of flows and of the relative radius $\bar{r}$ on the ratio of pressures p_1/p_2. In this case we increase $\bar{r}$ starting from unity and assign various values of C_p, C_q, and C_T.

An analysis of Eqs. (8)-(10) shows that if the values of dimensionless constants exceed minima which are equal to $C_\rho^{\min} \approx 1.1$, $C_q^{\min} \approx 0.2$; $C_T^{\min} \approx 0.5$, then $p_1/p_2 \geq 1$. Thus, the obtained minimum values of dimensionless constants determine the conditions of the origin of a near-wall flow.

Figure 2 presents the form of the dependences $p_1/p_2 = f(\bar{r}, \text{const})$, which have a maximum near $\bar{r} = 2$ and from the analysis of which it follows that the descending branches of the curves should be excluded from consideration, since pressure cannot decrease opposite to flow.

The radial position of the maximum determines the region of near-wall flow propagation, which for different conditions does not exceed $\bar{r} = 2-4$. It follows from analysis of the relations between p_1/p_2 and r_1/r_2 that the considered flows can have large pressure gradients.

The results obtained make it possible to estimate the mean flow velocity, which for an isothermal flow is equal to

$$V_2 = \sqrt{\left(\frac{4C_T}{C_f/b + 2} RT\right)}. \tag{11}$$

Since $C_f/b > 0$, then $4C_T/(C_f/b + 2)$ is also positive. Assuming $C_f/b \approx 1$, and $C_T = C_T^{\min}$ we find that $4C_T^{\min}/(C_f/b + 2) < k$ (where $k = 1.4$ for air). Thus, an isothermal flow can appear as a subsonic one.

The result of possible flow velocities agrees with the data of [4], which considers the plane adiabatic and potential flow of an ideal gas on a plane in the presence of a source (sink). It follows from the solution of this problem that this flow can be either subsonic or supersonic.

The obtained results are suitable for estimation of the effect of different factors on a secondary flow on a plane, which are necessary, for example, for practical investigations.

NOTATION

Re, Reynolds number; C_f, coefficient of friction resistance; a, flow acceleration; n, polytrope index; k, adiabatic index; r, radius; $\bar{r}$, relative radius; ρ, density; R, specific gas constant; μ, dynamic viscosity; ν, kinematic viscosity. Subscripts: fr, friction; in, inertia, f, fluid.

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